

AFFINE STRUCTURES ON QUANTUM PRINCIPAL BUNDLES

MICHIO ĐURĐEVICH

ABSTRACT. Quantum affine bundles are quantum principal bundles with affine quantum structure groups. A general theory of quantum affine bundles is presented. In particular, a detailed analysis of differential calculi over these bundles is performed, including the description of a natural differential calculus over the structure affine quantum group. A particular attention is given to the study of the specific properties of quantum affine connections and several purely quantum phenomena appearing in the context of quantum affine bundles. Various interesting constructions are presented. In particular, the main ideas are illustrated within the example of the quantum Hopf fibration.

1. INTRODUCTION

The aim of this paper is to study affine quantum principal bundles. These objects are noncommutative-geometric [1] analogs of classical affine extensions of principal bundles [11]. Quantum principal bundles [2, 3] generalize classical principal bundles—quantum groups play the role of the structure groups, and both the base manifold and the bundle are considered as general quantum objects. In this paper we shall consider very special quantum principal bundles, possessing *inhomogeneous* quantum structure groups. These groups are constructed by applying a general quantum affine extension procedure [13] to a given (compact matrix) quantum group.

The motivation for this work comes from a high importance of the classical affine bundles in various areas of standard differential geometry and theoretical physics. From geometrical point of view, the formalism of affine bundles gives a unified and powerful language to express properties and mutual interrelations of entities associated to the standard geometrical structures (as for example Riemannian/spin or symplectic manifolds and torsion structures). As a basic example coming from theoretical physics, let us mention various geometric formulations of classical general relativity as a gauge theory associated to Poincaré group. Hence, it looks promising to develop a quantum version of the theory of affine bundles. In particular, it is interesting to see how these objects are related with geometrical structures on quantum spaces, described by quantum frame bundles [5]. The formalism developed here may be used in constructing a noncommutative-geometric version of the gauge theory of gravity.

The paper is structured in the following way. In the next section, we shall start from a quantum principal bundle P , with the structure group G and over a given quantum space M . We shall then construct in a natural way another quantum principal bundle \tilde{P} over the same quantum space M , with the structure group \tilde{G} , which is the affine extension [13] of G relative to a given bicovariant bimodule Ψ over G . The left-invariant part of Ψ represents ‘translational’ degrees of freedom. After presenting the construction of \tilde{P} , we shall consider questions

related to differential calculus on affine bundles, following a general constructive approach to differential calculus developed in [6]. Our starting point will be the appropriate graded \ast -algebra \mathfrak{hor}_P , playing the role of the horizontal forms on the bundle P . This algebra will be further naturally extended to an affine horizontal forms algebra $\mathfrak{h}[\tilde{P}]$, representing horizontal forms on \tilde{P} . We shall then consider abstract ‘covariant derivative’ operators acting in both horizontal forms algebras, and study their mutual relations. Starting from such operators, it is possible to construct, in a natural manner, a differential calculus on the affine bundle \tilde{P} , as well as the associated differential (bicovariant \ast -) calculus Υ on \tilde{G} . A special attention will be given to the study of the interrelations between differential structures on P and \tilde{P} , and to the explicit description of the mentioned calculus Υ over \tilde{G} in terms of the calculus Γ over G and the initial bimodule Ψ . As we shall see, an interesting purely quantum phenomena appears—the left-invariant part of Υ is not a simple direct sum of the left-invariants Ψ_{inv} and Γ_{inv} . Instead, it will contain polynomial components over Ψ_{inv} , generally with arbitrary high degrees. This is a consequence of the fact that bicovariant bimodules are inherently braided. We shall also analyze the structure of affine connections, and the associated curvature, in light of the relations between the calculi on P and its affine extension \tilde{P} .

In the last section, concluding remarks and examples are made. At first, we shall consider quantum frame bundles, and study the specific properties of their affine extensions. Quantum frame bundles [5] are noncommutative-geometric counterparts of the classical frame bundles [11], incorporating into the quantum context the concept of a geometrical structure on a space (like metrics, spinor, symplectic and complex), via the system of special ‘coordinate 1-forms’. As we shall see, affine extensions of quantum frame bundles always include (as a subalgebra) a full differential calculus on the associated quantum plane.

As a simple but very instructive example, we shall illustrate all the basic elements of the theory on the quantum Hopf fibration—which is a quantum $U(1)$ -bundle over a quantum sphere [12], the total space P of which is a quantum $SU(2)$ group [15]. If we define Ψ to be a natural 2-dimensional bicovariant bimodule over $U(1)$, coming from the $3D$ -calculus [15] on the quantum $SU(2)$ group, then the affine extension turns out to be a deformation of the standard spin 2-covering of $E(2)$. The corresponding first-order calculus Υ will be computed explicitly. Surprisingly, in the truly quantum case this calculus is infinite-dimensional.

We shall also point out some analogies between the formalism of affine bundles and the formulation of differential calculus [3] for general quantum principal bundles.

2. AFFINE QUANTUM PRINCIPAL BUNDLES

This section is devoted to the construction of quantum principal affine bundles, naturally associated to a given quantum principal bundle P . The structure group of these bundles is the quantum [13] affine extension \tilde{G} of the structure quantum group G for P , with respect to a given bicovariant bimodule Ψ .

2.1. The Level of Groups

At first, let us recall the construction of \tilde{G} . Conceptually, we follow [13]. Let us consider the tensor bundle algebra $T(\Psi) = \Psi^{\otimes}$. The coproduct map admits a natural extension to a coassociative unital homomorphism $\phi: T(\Psi) \rightarrow T(\Psi) \otimes T(\Psi)$.

This extension is specified by

$$\phi(\vartheta) = \ell_\Psi(\vartheta) + \wp_\Psi(\vartheta)$$

where $\ell_\Psi: \Psi \rightarrow \mathcal{A} \otimes \Psi$ and $\wp_\Psi: \Psi \rightarrow \Psi \otimes \mathcal{A}$ are the left and the right action maps respectively. The extended counit $\epsilon: T(\Psi) \rightarrow \mathbb{C}$ is given by setting $\epsilon(\vartheta) = 0$, for each $\vartheta \in \Psi$. Finally, the antipode map $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ admits a unique antimultiplicative extension $\kappa: T(\Psi) \rightarrow T(\Psi)$, such that

$$\begin{array}{ccc} \Psi & \longrightarrow & \mathcal{A} \otimes \Psi \otimes \mathcal{A} \\ -\kappa \downarrow & & \downarrow \kappa \otimes \text{id} \otimes \kappa \\ \Psi & \longleftarrow & \mathcal{A} \otimes \Psi \otimes \mathcal{A} \end{array}$$

where the horizontal arrows represent the corresponding twofold coaction and bi-module multiplication respectively.

It is easy to see that $T(\Psi)$, endowed with the constructed maps, becomes a Hopf algebra. Furthermore, if Ψ is in addition a $*$ -covariant bimodule then we have the associated $*$ -involution $*$: $\Psi \rightarrow \Psi$ such that ℓ_Ψ and \wp_Ψ are hermitian. The algebra $T(\Psi)$ is equipped with the induced $*$ -structure, and in particular the extended coproduct map $\phi: T(\Psi) \rightarrow T(\Psi) \otimes T(\Psi)$ will be hermitian. The hermicity of the coproduct implies that ϵ is hermitian and that the composition map $*\kappa$ is involutive. In other words, $T(\Psi)$ is a Hopf $*$ -algebra. In what follows we shall assume that Ψ is equipped with a $*$ -structure.

Let us denote by $\Psi_{inv} = \mathbb{V}$ the left-invariant part of Ψ . This space is equipped with a natural right \mathcal{A} -comodule structure $\varkappa: \mathbb{V} \rightarrow \mathbb{V} \otimes \mathcal{A}$ and with a right \mathcal{A} -module structure $\circ: \mathbb{V} \otimes \mathcal{A} \rightarrow \mathbb{V}$. Here \varkappa is the restriction of the right-action on Ψ , and \circ is given by the formula

$$\{ \} \circ a = \kappa(a^{(1)}) \{ \} a^{(2)}.$$

Furthermore, the space \mathbb{V} is $*$ -invariant, and the following compatibility relations hold

$$\begin{aligned} \varkappa(\theta \circ a) &= \sum_k (\theta_k \circ a^{(2)}) \otimes \kappa(a^{(1)}) c_k a^{(3)} \\ \varkappa* &= (* \otimes *) \varkappa \quad (\theta \circ a)^* = \theta^* \circ \kappa(a)^*. \end{aligned}$$

where $\varkappa(\theta) = \sum_k \theta_k \otimes c_k$. All these maps admit natural extensions to the tensor algebra \mathbb{V}^\otimes , and the same compatibility relations hold for the extended maps (which for simplicity will be denoted by the same symbols). The extensions are fixed by

$$\begin{aligned} \varkappa(\theta\eta) &= \varkappa(\theta)\varkappa(\eta) \quad (\theta\eta)^* = \eta^*\theta^* \\ (\theta\eta) \circ a &= (\theta \circ a^{(1)})(\eta \circ a^{(2)}) \quad 1 \circ a = \epsilon(a)1. \end{aligned}$$

The tensor algebra $\mathbb{V}^\otimes \subseteq T(\Psi)$ is right ϕ -invariant, in the sense that

$$\phi[\mathbb{V}^\otimes] \subseteq \mathbb{V}^\otimes \otimes T(\Psi).$$

Let $\tau: \Psi \otimes_{\mathcal{A}} \Psi \rightarrow \Psi \otimes_{\mathcal{A}} \Psi$ be the braid operator, naturally associated [17] to the bicovariant bimodule Ψ . This operator is left/right covariant, and in particular it is completely determined by its restriction

$$\tau: \mathbb{V}^{\otimes 2} \rightarrow \mathbb{V}^{\otimes 2}, \quad \tau(\eta \otimes \theta) = \sum_k \theta_k \otimes (\eta \circ c_k),$$

where $\sum_k \theta_k \otimes c_k = \varkappa(\theta)$.

Now let us consider the τ -symmetrizer maps $Y_k: T(\Psi)^k \rightarrow T(\Psi)^k$, defined by

$$Y_k = \sum_{\pi \in S_k} \tau_\pi,$$

where τ_π is the operator obtained by replacing the transpositions figuring in any minimal decomposition of the permutation π by the corresponding τ -twists (this definition is consistent, due to the braid equation—in other words τ_π does not depend of the choice of minimal decompositions). The following factorizations hold:

$$Y_{k+l} = Y_{kl}(Y_k \otimes Y_l) = (Y_k \otimes Y_l)M_{kl},$$

where

$$Y_{kl} = \sum_{\pi \in S_{kl}} \tau_\pi \quad M_{kl} = \sum_{\pi \in S_{kl}} \tau_{\pi^{-1}},$$

and $S_{kl} \subset S_{k+l}$ consists of permutations preserving the orders of the first k and the last l factors.

In particular, it follows that the space $\ker(Y)$ is a two-sided ideal in $T(\Psi)$, where $Y: T(\Psi) \rightarrow T(\Psi)$ is the corresponding ‘total symmetrizer’. Furthermore, the following commutation relations hold:

$$*\tau* = \tau^{-1} \quad \kappa\tau = \tau\kappa.$$

This implies that $\ker(Y)^* = \ker(Y)$ and $\kappa[\ker(Y)] = \ker(Y)$.

We have a natural identification $\Psi \leftrightarrow \mathcal{A} \otimes \mathbb{V}$, and in particular $T(\Psi) \leftrightarrow \mathcal{A} \otimes \mathbb{V}^\otimes$. In terms of this identification, the braid τ and the extended coproduct ϕ are connected by the formula

$$(1) \quad \phi(a \otimes \vartheta) = \phi(a) \sum_{k+l=n} (\varkappa_k \otimes \text{id}^l) M_{kl}(\vartheta).$$

It follows that

$$\phi[\ker(Y)] \subseteq \ker(Y) \otimes T(\Psi) + T(\Psi) \otimes \ker(Y).$$

In other words, $\ker(Y) \subset T(\Psi)$ is a Hopf $*$ -ideal and therefore we can pass to the factor Hopf $*$ -algebra

$$\tilde{\mathcal{A}} = T(\Psi) / \ker(Y).$$

We shall denote by the same symbols $\phi, \epsilon, \kappa, *$ the corresponding projected maps.

The ideal $\ker(Y)$ is bicovariant. In particular, $\tilde{\mathcal{A}}$ is bicovariant and so we have the following natural decompositions:

$$\ker(Y) = \mathcal{A} \otimes \ker(Y \upharpoonright \Psi_{inv}^\otimes) \quad \tilde{\mathcal{A}} \leftrightarrow \mathcal{A} \otimes S(\mathbb{V}),$$

with an independent definition

$$S(\mathbb{V}) = \Psi_{inv}^\otimes / \ker(Y \upharpoonright \Psi_{inv}^\otimes) \\ S(\mathbb{V}) \leftrightarrow \text{im}(Y) \hookrightarrow \mathbb{V}^\otimes.$$

We shall also denote by the symbols \varkappa and \circ the corresponding right coaction and the right \mathcal{A} -module structure on $S(\mathbb{V})$ respectively. We have

$$\phi[S(\mathbb{V})] \subseteq S(\mathbb{V}) \otimes \tilde{\mathcal{A}},$$

and we shall denote by the symbol $\hat{\varkappa}: S(\mathbb{V}) \rightarrow S(\mathbb{V}) \otimes \tilde{\mathcal{A}}$ the corresponding restriction map.

2.2. The Level of Bundles

Let M be a quantum space and $P = (\mathcal{B}, i, F)$ a quantum principal G -bundle [3] over M . Here \mathcal{B} is a $*$ -algebra representing the quantum space P , and $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ is a counital coassociative $*$ -homomorphism playing the role of the dualized right action of G on P . Furthermore $i: \mathcal{V} \rightarrow \mathcal{B}$ is a $*$ -isomorphism between a $*$ -algebra \mathcal{V} representing the quantum space M and the F -fixed point subalgebra of \mathcal{B} . Therefore we can identify \mathcal{V} with its image in \mathcal{B} . Finally, we require that the action F is free, in the sense that a map $X: \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ given by $X(q \otimes b) = qF(b)$ is surjective.

We are now going to extend the bundle P , by including ‘translational’ degrees of freedom. This will give us a quantum principal \tilde{G} -bundle \tilde{P} , over the same quantum space M .

At first, let us define a $*$ -algebra $\tilde{\mathcal{B}}$ as

$$(2) \quad \tilde{\mathcal{B}} = \mathcal{B} \otimes S(\mathbb{V}),$$

at the level of vector spaces, and let us assume that $\tilde{\mathcal{B}}$ is equipped with a $*$ -algebra structure specified by cross-product type formulae

$$(3) \quad (q \otimes \theta)(b \otimes \eta) = \sum_k q b_k \otimes (\theta \circ c_k) \eta$$

$$(4) \quad (b \otimes \theta)^* = \sum_k b_k^* \otimes (\theta^* \circ c_k^*),$$

where $F(b) = \sum_k b_k \otimes c_k$. We see that both \mathcal{B} and $S(\mathbb{V})$ are understandable as $*$ -subalgebras of $\tilde{\mathcal{B}}$. The algebra $\tilde{\mathcal{B}}$ is naturally graded, with the grading induced from $S(\mathbb{V})$.

The maps F and $\hat{\varkappa}$ admit a natural common extension to a $*$ -homomorphism $H: \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}} \otimes \tilde{\mathcal{A}}$. More precisely H is the direct product of actions F and $\hat{\varkappa}$, and in particular

$$(5) \quad (H \otimes \text{id})H = (\text{id} \otimes \phi)H \quad (\text{id} \otimes \epsilon)H = \text{id}.$$

To prove that H is really a $*$ -homomorphism, it is sufficient to check that the commutation relations between \mathcal{B} and $S(\mathbb{V})$ are preserved. We have

$$\begin{aligned} \vartheta b &= \sum_k b_k (\vartheta \circ c_k) \longrightarrow \sum_{kl} (b_k \otimes c_k^{(1)}) \left\{ (\vartheta_l \circ c_k^{(3)}) \otimes \kappa(c_k^{(2)}) d_l \kappa(c_k^{(4)}) \right\} \\ &= \sum_{kl} b_k (\vartheta_l \circ c_k^{(1)}) \otimes d_l c_k^{(2)} = \sum_{kl} (\vartheta_l \otimes d_l) (b_k \otimes c_k) = H(\vartheta)H(b), \end{aligned}$$

where $\varphi \in \mathcal{B}$ and $\vartheta \in S(\mathbb{V})$ with $\sum_l \vartheta_l \otimes d_l = \hat{\varkappa}(\vartheta)$. In other words, \tilde{G} acts by ‘automorphisms’ on $\tilde{\mathcal{B}}$.

Lemma 2.1. *The fixed-point subalgebra for the total action H on $\tilde{\mathcal{B}}$ coincides with the algebra $\mathcal{V} \leftrightarrow \mathcal{V} \otimes \mathbb{C}$.*

Proof. Let us consider an arbitrary element $q = \sum_{b\vartheta} b \otimes \vartheta$, with linearly independent and homogeneous $\vartheta \in S(\mathbb{V})$ and arbitrary $b \in \mathcal{B}$. From the form of the action $\hat{\varkappa}$, it follows that

$$H(q) = \sum_{b\vartheta}^* F(b)\vartheta + q'$$

where the summation is performed over pairs (b, ϑ) having the maximal degree, and $q' \in \tilde{\mathcal{B}} \otimes \tilde{\mathcal{A}}$ consists of summands with lower degrees (here we have compared degrees relative to the second tensoriand). It follows that

$$(6) \quad \mathcal{B} = H^{-1} \left\{ \tilde{\mathcal{B}} \otimes \mathcal{A} \right\}.$$

In particular q is H -invariant iff $q = b \otimes 1 \leftrightarrow b$, and b is F -invariant. \square

As already mentioned, the freeness of the action F is expressed as the surjectivity of the natural map $X: \mathcal{B} \otimes_{\mathbb{V}} \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$. As explained in [7], the map X is also injective for compact structure groups G , so that P defines a Hopf-Galois extension [14]. In particular, we can introduce the quantum ‘translation map’ by inverting X over \mathcal{A} —in other words we define $\tau: \mathcal{A} \rightarrow \mathcal{B} \otimes_{\mathbb{V}} \mathcal{B}$ by

$$\tau(a) = X^{-1}(1 \otimes a),$$

so that

$$b \otimes a = X[b\tau(a)] \quad \forall b \in \mathcal{B}, a \in \mathcal{A}.$$

If G is a general non-compact structure group then the existence of τ is a non-trivial condition on the bundle P . However, the main constructions of this paper will still work if we assume that τ exists (that is, if P is a Hopf-Galois extension—equivalent to the bijectivity of X).

Now we shall prove that τ admits a natural extension $\tau: \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{B}} \otimes_{\mathbb{V}} \tilde{\mathcal{B}}$, which is actually the affine translation map. In what follows, we shall also use the symbolic notation $\tau() = []_1 \otimes []_2$.

We shall start by writing the explicit formula for the extended translation map. Let $\tau: \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{B}} \otimes_{\mathbb{V}} \tilde{\mathcal{B}}$ be a linear map defined by

$$(7) \quad \tau(a \otimes \vartheta) = \sum_{i \geq 0} \sum_{\alpha, \beta \in S[i]} (-1)^i I\{t_{\beta i}\} [c_{\beta \alpha}^i]_1 \tau(a) [c_{\beta \alpha}^i]_2 \vartheta_{\alpha i}.$$

In the above formula $\vartheta \in S(\mathbb{V})^n$ and the elements $\{t_{\alpha i} \mid \alpha \in S[i]\}$ form a basis in the space $S(\mathbb{V})^i$. Furthermore, the map $I: S(\mathbb{V}) \rightarrow S(\mathbb{V})$ is the canonical ‘total braided inverse’. More precisely, this map is originally defined on \mathbb{V}^{\otimes} as the unital grade-preserving map totally reversing the order with the help of the braiding τ —and then it is projected down to $S(\mathbb{V})$. Finally, we have put

$$\begin{aligned} \varkappa(t_{\alpha i}) &= \sum_{\beta \in S[i]} t_{\beta i} \otimes c_{\beta \alpha}^i \\ \sum_{\alpha \in S[i]} t_{\alpha i} \otimes \vartheta_{\alpha i} &= M_{in-i}(\vartheta). \end{aligned}$$

The consistency of the above formula follows from the fact that operators M_{in-i} are projectable down to $S(\mathbb{V})^n$.

In particular, we see that

$$(8) \quad \tau \lceil \mathcal{A} = \underbrace{\tau: \mathcal{A} \rightarrow \mathcal{B} \otimes_{\mathbb{V}} \mathcal{B}}_{\text{the original map}} \quad \tau(\theta) = - \sum_k \theta_k \tau(c_k) + 1 \otimes \theta,$$

where $\theta \in \mathbb{V}$ and $\sum_k \theta_k \otimes c_k = \varkappa(\theta)$.

Lemma 2.2. *The extended map satisfies*

$$\begin{aligned} [\psi]_1 \otimes H[\psi]_2 &= [\psi^{(1)}]_1 \otimes [\psi^{(1)}]_2 \otimes \psi^{(2)} \\ [\psi]_1 [\psi]_2 &= \epsilon(\psi) 1 \\ {}^{\text{op}}H[\psi]_1 \otimes [\psi]_2 &= \psi^{(1)} \otimes [\psi^{(2)}]_1 \otimes [\psi^{(2)}]_2 \end{aligned}$$

where ${}^{\text{op}}H: \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{A}} \otimes \tilde{\mathcal{B}}$ is the opposite action, defined by ${}^{\text{op}}H = (\kappa^{-1} \otimes \text{id})(\circ)H$.

Proof. These identities follow by direct calculations, applying formula (1) for the extended coproduct and the antipode formula

$$(9) \quad \kappa(a \otimes \vartheta) = \left\{ (-1)^{\partial \vartheta} \sum_k \text{I}\{\vartheta_k\} \kappa(c_k) \right\} \kappa(a)$$

which holds both in $\tilde{\mathcal{A}}$ and $\text{T}(\Psi)$. Finally, let us observe that the original translation map $\tau: \mathcal{A} \rightarrow \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B}$ satisfies our identities by construction. \square

The above lemma implies that the extended $X: \tilde{\mathcal{B}} \otimes_{\mathcal{V}} \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}} \otimes \tilde{\mathcal{A}}$ is also bijective, and that we have

$$X^{-1}(1 \otimes \psi) = \tau(\psi).$$

In other words $\tilde{\mathcal{B}}$ equipped with H determines a Hopf-Galois extension and the extended τ is the affine translation map. In particular, the action H is free, and we have a quantum principal \tilde{G} -bundle $\tilde{P} = (\tilde{\mathcal{B}}, i, H)$ over the quantum space M .

Definition 1. The bundle \tilde{P} is called *the quantum affine extension* of the quantum principal bundle P .

We can interpret the quantum space \tilde{P} as a bundle over P . The fibering of \tilde{P} over P is trivial, and the fiber is the quantum affine space associated to \mathbb{V} and τ —described by the symmetric algebra $S(\mathbb{V})$. The algebra \mathcal{B} is the fixed-point subalgebra for the action $\text{id} \otimes \hat{\kappa}: \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}} \otimes \tilde{\mathcal{A}}$. Let $p_*: \tilde{\mathcal{B}} \rightarrow \mathcal{B} = \tilde{\mathcal{B}}^0$ be the canonical projection map. If we interpret F as the restricted action of \tilde{G} we can write

$$Fp_* = (p_* \otimes \text{id})H,$$

and therefore \tilde{P} is geometrically interpretable as the extension of the bundle P .

As explained in [9], if a quantum principal bundle admits the translation map then there exists an intrinsic braid operator, twisting the functions of the bundle. In our context this means that we can introduce a braiding $\sigma_M: \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \rightarrow \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B}$, and its affine extension $\sigma_M: \tilde{\mathcal{B}} \otimes_{\mathcal{V}} \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}} \otimes_{\mathcal{V}} \tilde{\mathcal{B}}$.

Lemma 2.3. *There exists an intrinsic braiding $\sigma_M: \tilde{\mathcal{B}} \otimes_{\mathcal{V}} \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}} \otimes_{\mathcal{V}} \tilde{\mathcal{B}}$, extending the initial braiding σ_M . We have*

$$(10) \quad \sigma_M(\theta \otimes \varphi) = \sum_k [\theta_k, \varphi] \tau(c_k) + \varphi \otimes \theta.$$

for each $\varphi \in \tilde{\mathcal{B}}$ and $\theta \in \mathbb{V}$.

Proof. According to [9], the braidings are defined by the formula

$$(11) \quad \sigma_M(b \otimes \psi) = \sum_k b_k \psi[c_k]_1 \otimes [c_k]_2,$$

where $\sum_k b_k \otimes c_k = H(b)$. Taking this into account, formula (10) directly follows from (8). \square

Let us observe that formula (10) uniquely determines the extended braiding σ_M . This follows from the fact that the braiding is always functorial relative to the product in the bundle algebra [9]. Explicitly, the formula (10) is generalized to the following expression describing the braiding between $S(\mathbb{V})$ and $\tilde{\mathcal{B}}$ (the notation is the same as in the definition of the affine translation map):

Proposition 2.4. *We have*

$$(12) \quad \sigma_M(\vartheta \otimes \varphi) = \varphi \otimes \vartheta + \sum_{i>0} \sum_{\alpha, \beta \in S[i]} \{t_{\beta i}|\varphi\}_\tau [c_{\beta\alpha}^i]_1 \otimes [c_{\beta\alpha}^i]_2 \vartheta_{\alpha i},$$

where we have introduced ‘braided multicommutators’ $\{|\}$ defined by

$$(13) \quad \{\vartheta|\varphi\}_\tau = \sum_{i \geq 0} \sum_{\alpha \in S[i]} (-1)^{n-i} t_{\alpha i} \varphi I\{\vartheta_{\alpha i}\}.$$

Proof. This identity follows directly from (11) and (7). It is also possible to derive (12) from (10), inductively applying the ‘left variant’ of the mentioned functoriality property. \square

3. AFFINE HORIZONTAL FORMS & RELATED CALCULUS

3.1. Horizontal Forms

In this subsection we shall refine our construction of the affine bundles, in order to include the analogs of horizontal forms in the game. This is the first step towards constructing the appropriate complete differential calculus on affine quantum principal bundles.

Let us consider a quantum principal G -bundle $P = (\mathcal{B}, i, F)$ over a quantum space M . Let \mathfrak{hor}_P be graded $*$ -algebra such that $\mathcal{B} = \mathfrak{hor}_P^0$, equipped with a coassociative grade-preserving $*$ -homomorphism $F^\wedge: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P \otimes \mathcal{A}$ extending the action F . Let $\Omega_M \subseteq \mathfrak{hor}_P$ be the corresponding fixed-point $*$ -subalgebra. Let us assume that Ω_M is equipped with a hermitian first-order differential $\overset{M}{d}: \Omega_M \rightarrow \Omega_M$. We shall interpret the elements of \mathfrak{hor}_P as ‘abstract horizontal forms’. In the framework of this interpretation, the elements of Ω_M naturally correspond to the forms on the base.

Let $\mathfrak{zh}_P \subseteq \mathfrak{hor}_P$ be the graded commutant of Ω_M in \mathfrak{hor}_P . This is a graded $*$ -subalgebra of \mathfrak{hor}_P , invariant under the action F^\wedge . As explained in [6], the formula

$$\xi \circ a = [a]_1 \xi [a]_2,$$

consistently defines a right \mathcal{A} -module structure in the space \mathfrak{zh}_P . This map, together with F^\wedge and $*$, determines a bicovariant $*$ -bimodule structure over G . The following commutation relation holds:

$$(14) \quad \xi \varphi = (-1)^{\partial \varphi \partial \xi} \sum_k \varphi_k (\xi \circ c_k),$$

where $\varphi \in \mathfrak{hor}_P$. The algebra \mathfrak{zh}_P is braided-commutative, in a natural manner.

Let us now prove a useful technical lemma.

Lemma 3.1. *Let $\Delta: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ be an arbitrary graded derivation, intertwining the map F^\wedge . Then Δ is reduced in the space \mathfrak{zh}_P , and*

$$(15) \quad \Delta(\xi \circ a) = \Delta(\xi) \circ a,$$

for each $\xi \in \mathfrak{zh}_P$ and $a \in \mathcal{A}$.

Proof. The fact that Δ is reduced in Ω_M , together with the graded Leibniz rule and the definition of \mathfrak{zh}_P , implies that Δ is reduced in \mathfrak{zh}_P . We have

$$\begin{aligned}\Delta(\xi \circ a) &= (\Delta[a]_1)\xi[a]_2 + (-1)^{\partial\xi\partial\Delta}[a]_1\xi\Delta[a]_2 + [a]_1\Delta(\xi)[a]_2 \\ &= \Delta[a^{(1)}]_1[a^{(1)}]_2(\xi \circ a^{(2)}) \\ &\quad + [a^{(1)}]_1\Delta[a^{(1)}]_2(\xi \circ a^{(2)}) + \Delta(\xi) \circ a \\ &= \Delta\{[a^{(1)}]_1[a^{(1)}]_2\}(\xi \circ a^{(2)}) + \Delta(\xi) \circ a = \Delta(\xi) \circ a,\end{aligned}$$

because of the identity $[a]_1[a]_2 = \epsilon(a)1$. \square

Let us denote by $\mathfrak{der}(P)$ the real affine space of all hermitian F^\wedge -covariant first-order antiderivations $D: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ extending the differential $\overset{\mathcal{H}}{d}: \Omega_M \rightarrow \Omega_M$. In what follows, it will be assumed that $\mathfrak{der}(P) \neq \emptyset$. In accordance with [6], we adopt the following

Definition 2. The elements of $\mathfrak{der}(P)$ are called *bundle derivatives* for P . The associated vector space $\overrightarrow{\mathfrak{der}}(P)$ consists of hermitian first-order right-covariant antiderivations E on \mathfrak{hor}_P which vanish on Ω_M .

We shall interpret the elements of $\mathfrak{der}(P)$ as covariant derivative maps for the bundle P . This interpretation is justified after constructing the appropriate complete calculus on the bundle, such that bundle derivatives are in 1–1 correspondence with covariant derivatives of regular connections [3]. The construction of the complete calculus on the bundle is presented in full details in [6].

As explained in [6], to each $D \in \mathfrak{der}(P)$ we can intrinsically associate a linear map $\varrho_D: \mathcal{A} \rightarrow \mathfrak{hor}_P$, with the help of the formula

$$(16) \quad D^2(\varphi) = - \sum_k \varphi_k \varrho_D(c_k),$$

where $\sum_k \varphi_k \otimes c_k = F^\wedge(\varphi)$. This map plays the role of *the curvature* of D . It is uniquely defined by D . Moreover, the following equalities hold

$$(17) \quad F^\wedge \varrho_D(a) = (\varrho_D \otimes \text{id})\text{ad}(a)$$

$$(18) \quad D\varrho_D(a) = 0$$

$$(19) \quad \varrho_D[\kappa(a)^*] = -\varrho_D(a)^*$$

$$(20) \quad \varrho_D(a)\varphi = \sum_k \varphi_k \varrho_D(ac_k)$$

for each $a \in \ker(\epsilon)$ and $\varphi \in \mathfrak{hor}_P$.

Furthermore, for each $E \in \overrightarrow{\mathfrak{der}}(P)$ there exists a unique $\chi_E: \mathcal{A} \rightarrow \mathfrak{hor}_P$ such that

$$(21) \quad E(\varphi) = -(-1)^{\partial\varphi} \sum_k \varphi_k \chi_E(c_k),$$

for each $\varphi \in \mathfrak{hor}_P$. The following equalities hold:

$$(22) \quad F^\wedge \chi_E(a) = (\chi_E \otimes \text{id})\text{ad}(a)$$

$$(23) \quad \chi_E[\kappa(a)^*] = -\chi_E(a)^*$$

$$(24) \quad \chi_E(a)\varphi = (-1)^{\partial\varphi} \sum_k \varphi_k \chi_E(ac_k)$$

for each $a \in \ker(\epsilon)$ and $\varphi \in \mathfrak{hor}_P$. Let us also observe that

$$\varrho_D(1) = \chi_E(1) = 0.$$

Applying the construction from the previous section to the pair $(\mathfrak{hor}_P, F^\wedge)$ we obtain a $*$ -algebra

$$\mathfrak{h}[\tilde{P}] \leftrightarrow \mathfrak{hor}_P \otimes S(\mathbb{V}),$$

where the product and the conjugation are given by the same cross-product type rules, together with the action $H: \mathfrak{h}[\tilde{P}] \rightarrow \mathfrak{h}[\tilde{P}] \otimes \tilde{\mathcal{A}}$. We have the following interesting characterization

$$\mathfrak{hor}_P = \mathfrak{hor}_P \otimes \mathbb{C} = H^{-1}(\mathfrak{h}[\tilde{P}] \otimes \mathcal{A}).$$

The algebra $\mathfrak{h}[\tilde{P}]$ is bigraded, in a natural manner.

The elements of the constructed algebra $\mathfrak{h}[\tilde{P}]$ play the role of horizontal differential forms on the quantum affine bundle \tilde{P} . Restricted to \mathfrak{hor}_P , the action H coincides with F^\wedge . In particular, the H -fixed-point subalgebra of $\mathfrak{h}[\tilde{P}]$ coincides with Ω_M . We shall denote by the same symbol $p_*: \mathfrak{h}[\tilde{P}] \rightarrow \mathfrak{hor}_P$ the canonical projection map.

In a similar way, let us introduce the real affine space $\mathfrak{der}(\tilde{P})$ and its vectorization $\overrightarrow{\mathfrak{der}}(\tilde{P})$. The maps from $\mathfrak{der}(\tilde{P})$ play the role of affine covariant derivatives. Actually, this interpretation of $\mathfrak{der}(P)$ and $\mathfrak{der}(\tilde{P})$ is fully justified after constructing the intrinsic complete differential calculi over the bundles P and \tilde{P} . The maps from $\mathfrak{der}(P)$ and $\mathfrak{der}(\tilde{P})$ become ‘true’ covariant derivatives associated to corresponding regular connections, in the sense of [3].

Let us consider an arbitrary $D \in \mathfrak{der}(\tilde{P})$. The intertwining property implies

$$\begin{aligned} D(\mathfrak{hor}_P) &\subseteq \mathfrak{hor}_P \\ D(\mathbb{V}) &\subseteq \mathfrak{hor}_P. \end{aligned}$$

Moreover, because of the graded Leibniz rule, every such D is completely determined by its restrictions

$$(D|_{\mathfrak{hor}_P}) \in \mathfrak{der}(P), \quad (D|_{\mathbb{V}}) = \lambda: \mathbb{V} \rightarrow \mathfrak{hor}_P.$$

Proposition 3.2. (i) *Conversely, let us consider an arbitrary first-order linear map $\lambda: \mathbb{V} \rightarrow \mathfrak{hor}_P$ satisfying*

$$(25) \quad F^\wedge \lambda = (\lambda \otimes \text{id})\varkappa$$

$$(26) \quad \lambda(\theta)\varphi = (-1)^{\partial\varphi} \sum_k \varphi_k \lambda(\theta \circ c_k),$$

where $\varphi \in \mathfrak{hor}_P$ and $\sum_k \varphi_k \otimes c_k = F^\wedge(\varphi)$. Then every F^\wedge -covariant first-order antiderivation $D: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ admits a unique extension to a first-order antiderivation $D: \mathfrak{h}[\tilde{P}] \rightarrow \mathfrak{h}[\tilde{P}]$ such that $D(\theta) = \lambda(\theta)$, for each $\theta \in \mathbb{V}$.

(ii) *This map is automatically H -covariant. It is hermitian, if and only if both λ and the initial D are hermitian.*

Proof. At first, we have to check the compatibility of the definition of the extended covariant derivative, with the relations defining the algebra $\mathfrak{h}[\tilde{P}]$. A direct computation gives

$$\begin{aligned} 0 = \theta\varphi - \sum_k \varphi_k(\theta \circ c_k) &\longrightarrow \lambda(\theta)\varphi + \theta D(\varphi) \\ &\quad - \sum_k D(\varphi_k)(\theta \circ c_k) - (-1)^{\partial\varphi} \varphi_k \lambda(\theta \circ c_k) = 0, \end{aligned}$$

because of the F^\wedge -covariance of $D: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ and the definition of λ . Furthermore,

$$S(\mathbb{V})^n \ni \vartheta \longrightarrow (\{\}^\wedge \otimes \text{id}^{n-1}) M_{1n-1}(\vartheta)$$

which is consistent because of the invariance of $\ker(Y_n)$ under all operators M_{kl} , where $k+l=n$. Therefore, our definition together with the graded Leibniz rule consistently and uniquely determines the extension $D: \mathfrak{h}[\tilde{P}] \rightarrow \mathfrak{h}[\tilde{P}]$. Finally, to prove (ii) it is sufficient to check that \mathbb{V} is covariant under the action of $*$ and H . The hermicity is trivial, while

$$(D \otimes \text{id})H(\theta) = (D \otimes \text{id})[1 \otimes \theta + \varkappa(\theta)] = (\lambda \otimes \text{id})\varkappa(\theta) = HD(\theta),$$

for each $\theta \in \mathbb{V}$. This completes the proof. \square

Let us denote by Λ the graded space of all maps $\lambda: \mathbb{V} \rightarrow \mathfrak{hor}_P$ intertwining \varkappa and F^\wedge and satisfying

$$(27) \quad \lambda(\theta)\varphi = (-1)^{\partial\varphi\partial\lambda} \sum_k \varphi_k \lambda(\theta \circ c_k).$$

In particular, the above equality implies that every map $\lambda \in \Lambda$ takes the values from \mathfrak{zh}_P . Furthermore, (27) is equivalent to the intertwining property

$$(28) \quad \lambda(\theta \circ a) = \lambda(\theta) \circ a.$$

The formula

$$\lambda^*(\theta) = \lambda(\theta^*)^*$$

determines a natural conjugation on this space. According to the above analysis,

$$(29) \quad \mathfrak{der}(\tilde{P})_{\mathbb{C}} = \mathfrak{der}(P)_{\mathbb{C}} \oplus \Lambda^1,$$

which corresponds to the classical decomposition [11] of the affine connections.

Similarly, we can introduce the affine transition maps $E: \mathfrak{h}[\tilde{P}] \rightarrow \mathfrak{h}[\tilde{P}]$. They form a real affine space $\overrightarrow{\mathfrak{der}}(\tilde{P})$. The following natural decomposition holds

$$(30) \quad \overrightarrow{\mathfrak{der}}(\tilde{P})_{\mathbb{C}} = \overrightarrow{\mathfrak{der}}(P)_{\mathbb{C}} \oplus \Lambda^1,$$

in accordance with (29) and the definition of $\overrightarrow{\mathfrak{der}}(\tilde{P})$.

We are going to analyze relations between the affine curvature map (defined in the framework of \tilde{P}) and the standard curvature map (defined at the level of P).

Let $\varrho_D: \tilde{\mathcal{A}} \rightarrow \mathfrak{h}[\tilde{P}]$ be the curvature map associated to the affine covariant derivative D . It is the extension of the curvature $\varrho_D: \mathcal{A} \rightarrow \mathfrak{hor}_P$, associated to $D: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$.

Proposition 3.3. *We have*

$$(31) \quad \varrho_D(\theta) = - \sum_k \theta_k \varrho_D(c_k) - D\lambda(\theta),$$

where $\theta \in \mathbb{V}$ and $\sum_k \theta_k \otimes c_k = \varkappa(\theta)$.

Proof. Applying the definitions of D , ϱ_D and λ we obtain

$$D^2(\theta) = D\lambda(\theta) = -\varrho_D(\theta) - \sum_k \theta_k \varrho_D(c_k),$$

in other words (31) holds. \square

Similarly, the maps $E: \mathfrak{h}[\tilde{P}] \rightarrow \mathfrak{h}[\tilde{P}]$ are naturally determined by linear functionals $\chi_E: \tilde{\mathcal{A}} \rightarrow \mathfrak{h}[\tilde{P}]$. These maps extend previously considered $\chi_E: \mathcal{A} \rightarrow \mathfrak{hor}_P$, and we have

$$(32) \quad \chi_E(\theta) = - \sum_k \theta_k \chi_E(c_k) - E(\theta),$$

for each $\theta \in \mathbb{V}$.

It is worth noticing that principal relations (16)–(24) are preserved at the affine level. In particular, formulae (31)–(32) completely determine the affine extensions of the original maps ϱ_D and χ_E .

3.2. The Induced Affine Calculus

Let us consider the algebra \mathfrak{hor}_P . As explained in [6], the system of maps $\varrho_D, \chi_E: \mathcal{A} \rightarrow \mathfrak{hor}_P$ naturally determines a bicovariant *-calculus Γ over G . By definition, this calculus is based on the right \mathcal{A} -ideal $\mathcal{R} \subset \ker(\epsilon)$ consisting of the elements annihilated by all ϱ_D and χ_E . In order to get a nontrivial calculus, we shall assume that at least one bundle derivative has a non-zero curvature. Geometrically, Γ is the minimal calculus on G compatible with the internal structure of horizontal forms \mathfrak{hor}_P .

By construction, the maps ϱ_D and χ_E are factorizable through \mathcal{R} , and hence interpretable as $\varrho_D, \chi_E: \Gamma_{inv} \rightarrow \mathfrak{hor}_P$. Let us also observe that (20)/(24) imply that ϱ_D, χ_E take the values from \mathfrak{zh}_P , and that

$$(33) \quad \varrho_D(\xi \circ a) = \varrho_D(\xi) \circ a \quad \chi_E(\xi \circ a) = \chi_E(\xi) \circ a.$$

On the other hand, applied at the level of affine bundles, the mentioned construction of the structure group calculus gives us an intrinsic bicovariant *-calculus Υ over \tilde{G} . It is determined by the system of affine maps $\varrho_D, \chi_E: \tilde{\mathcal{A}} \rightarrow \mathfrak{h}[\tilde{P}]$.

Before passing to the explicit construction of the affine calculus Υ , let us write down some algebraic relations involving the maps ϱ_D and χ_E . We have already introduced the graded commutant \mathfrak{zh}_P . It is obviously included in the ‘affine’ graded commutant

$$\mathfrak{zh}[\tilde{P}] \leftrightarrow \mathfrak{zh}_P \otimes S(\mathbb{V})$$

which is a right $\tilde{\mathcal{A}}$ -module, in a natural manner (the module structure is constructed with the help of the affine translation map). Explicitly, the right $\tilde{\mathcal{A}}$ -module structure on $\mathfrak{zh}[\tilde{P}]$ is specified by

$$\begin{aligned} (\zeta \otimes \vartheta) \circ a &= (\zeta \circ a^{(1)}) \otimes (\vartheta \circ a^{(2)}) \\ \xi \circ \vartheta &= \sum_{i \geq 0} \sum_{\alpha, \beta \in S[i]} (-1)^i I_{\{t_{\beta i}\}} (\xi \circ c_{\beta \alpha}^i) \vartheta_{\alpha i}, \end{aligned}$$

where $\xi \in \mathfrak{zh}[\tilde{P}]$, $a \in \mathcal{A}$ and $\vartheta \in S(\mathbb{V})$.

The maps $\varrho_D, \chi_E: \tilde{\mathcal{A}} \rightarrow \mathfrak{h}[\tilde{P}]$ take their values from $\mathfrak{zh}[\tilde{P}]$. Furthermore, according to the general theory [6], the maps ϱ_D and χ_E , restricted to $\ker(\epsilon: \tilde{\mathcal{A}} \rightarrow \mathbb{C})$,

intertwine the right $\tilde{\mathcal{A}}$ -multiplication and the right $\tilde{\mathcal{A}}$ -module structure on $\mathfrak{zh}[\tilde{P}]$. Explicitly, we have

$$\begin{aligned}
 (34) \quad \varrho_D(\xi a) &= \varrho_D(\xi) \circ a = [a]_1 \varrho_D(\xi) [a]_2 & \chi_E(\xi a) &= \chi_E(\xi) \circ a \\
 \varrho_D(\xi \vartheta) &= \varrho_D(\xi) \circ \vartheta = \sum_{i \geq 0} \sum_{\alpha, \beta \in S[i]} (-1)^i \mathbf{I}\{t_{\beta i}\} \{ \varrho_D(\xi) \circ c_{\beta \alpha}^i \} \vartheta_{\alpha i} \\
 (35) \quad \chi_E(\xi \vartheta) &= \chi_E(\xi) \circ \vartheta = \sum_{i \geq 0} \sum_{\alpha, \beta \in S[i]} (-1)^i \mathbf{I}\{t_{\beta i}\} \{ \chi_E(\xi) \circ c_{\beta \alpha}^i \} \vartheta_{\alpha i},
 \end{aligned}$$

where the notation is the same as in formula (7), with $a \in \mathcal{A}$ and $\vartheta \in S(\mathbb{V})$ while $\xi \in \tilde{\mathcal{A}}$ satisfies $\epsilon(\xi) = 0$. In particular, for $\vartheta = \theta \in \mathbb{V}$ formulae (35) reduce to

$$\begin{aligned}
 (36) \quad \varrho_D(\xi \theta) &= \varrho_D(\xi) \circ \theta = \varrho_D(\xi) \theta - \sum_k \theta_k \{ \varrho_D(\xi) \circ c_k \} \\
 \chi_E(\xi \theta) &= \chi_E(\xi) \circ \theta = \chi_E(\xi) \theta - \sum_k \theta_k \{ \chi_E(\xi) \circ c_k \}.
 \end{aligned}$$

The above expressions allow us to construct explicitly the calculus Υ , in terms of the calculus Γ and the algebraic relations in $\mathfrak{h}[\tilde{P}]$. In particular, we see that the space Υ_{inv} contains in a natural manner Γ_{inv} and the image of \mathbb{V} . Moreover Υ_{inv} is built from Γ_{inv} and \mathbb{V} , by the right action of $S(\mathbb{V})$.

The presented construction of the calculus Υ can be easily abstracted from the context of quantum principal bundles. We are going to make a small digression from our main theme, and to present an abstract version of the construction of Υ .

Let us consider a $*$ -algebra \mathcal{H} , equipped with the right \mathcal{A} -module structure \circ and a right counital \mathcal{A} -comodule structure $\Delta: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{A}$, such that the standard compatibility properties

$$\begin{aligned}
 (\varphi \circ a)^* &= \varphi \circ \kappa(a)^* & \Delta* &= (* \otimes *) \Delta \\
 (\varphi \psi) \circ a &= (\varphi \circ a^{(1)}) (\psi \circ a^{(2)}) \\
 \Delta(\varphi \circ a) &= \sum_k (\varphi_k \circ a^{(2)}) \otimes \kappa(a^{(1)}) c_k a^{(3)}
 \end{aligned}$$

hold, where $\sum_k \varphi_k \otimes c_k = \Delta(\varphi)$.

Let us now introduce, starting from the algebra \mathcal{H} , a $*$ -algebra $\tilde{\mathcal{H}} = \mathcal{H} \otimes S(\mathbb{V})$, equipped with the standard cross-product structure. The maps \circ and Δ are naturally extendible to the right $\tilde{\mathcal{A}}$ -module structure \circ on $\tilde{\mathcal{H}}$ and the right comodule structure $\tilde{\Delta}: \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}} \otimes \tilde{\mathcal{A}}$, such that the above compatibility properties are preserved. Explicitly, the map $\tilde{\Delta}$ is constructed by combining Δ and $\hat{\varkappa}$. The extension of \circ is constructed in two steps. At first, right \mathcal{A} -module \circ -structures on \mathcal{H} and $S(\mathbb{V})$ naturally combine to a right \mathcal{A} -module structure on $\tilde{\mathcal{H}}$. Secondly, the formula

$$(37) \quad \xi \circ \vartheta = \sum_{i \geq 0} \sum_{\alpha, \beta \in S[i]} (-1)^i \mathbf{I}\{t_{\beta i}\} (\xi \circ c_{\beta \alpha}^i) \vartheta_{\alpha i},$$

uniquely and consistently defines an extension of \circ to a right $\tilde{\mathcal{A}}$ -module structure on $\tilde{\mathcal{H}}$. Here $\vartheta \in S(\mathbb{V})$ and the notation is the same as in formula (7).

Let us assume that a map $\pi: \mathcal{A} \rightarrow \mathcal{H}$ is given, such that

$$\begin{aligned} \Delta\pi &= (\pi \otimes \text{id})\text{ad} & \pi(a)^* &= -\pi[\kappa(a)^*] \\ \pi(ab) &= \pi(a) \circ b + \epsilon(a)\pi(b), & \Rightarrow & \pi(1) = 0. \end{aligned}$$

This map naturally defines a bicovariant *-calculus Γ over G . It is based on the right \mathcal{A} -ideal $\mathcal{R} = \ker(\epsilon) \cap \ker(\pi)$.

Furthermore we shall assume that a map $\zeta: \mathbb{V} \rightarrow \mathcal{H}$ is given, satisfying

$$\Delta\zeta = (\zeta \otimes \text{id})\varkappa, \quad \zeta* = *\zeta, \quad \zeta(\theta \circ a) = \zeta(\theta) \circ a.$$

We are going to construct, with the help of ζ , a natural extension $\hat{\pi}$ of π , which is defined on $\tilde{\mathcal{A}}$ and takes the values from $\tilde{\mathcal{H}}$. Firstly, let us define $(\hat{\pi} \upharpoonright \mathbb{V}): \mathbb{V} \rightarrow \tilde{\mathcal{H}}$ by

$$(38) \quad \hat{\pi}(\theta) = \zeta(\theta) - \sum_k \theta_k \pi(c_k),$$

where $\sum_k \theta_k \otimes c_k = \varkappa(\theta)$. The full map $\hat{\pi}: \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{H}}$ is defined by the following construction:

Proposition 3.4. *The formula*

$$(39) \quad \hat{\pi}(q\vartheta) = \sum_{i \geq 0} \sum_{\alpha, \beta \in S[i]} (-1)^i \mathbf{I}\{t_{\beta i}\} \{\hat{\pi}(q) \circ c_{\beta\alpha}^i\} \vartheta_{\alpha i}$$

consistently defines a map $\hat{\pi}: \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{H}}$, extending the map $\pi: \mathcal{A} \rightarrow \mathcal{H}$ and previously defined restriction $\hat{\pi} \upharpoonright \mathbb{V}$. Here $q \in \ker(\epsilon: \tilde{\mathcal{A}} \rightarrow \mathbb{C})$ and $\vartheta \in S(\mathbb{V})$. The following identities hold:

$$(40) \quad \hat{\Delta}\hat{\pi} = (\hat{\pi} \otimes \text{id})\text{ad} \quad \hat{\pi}(\xi)^* = -\hat{\pi}[\kappa(\xi)^*]$$

$$(41) \quad \hat{\pi}(\xi\psi) = \hat{\pi}(\xi) \circ \psi + \epsilon(\xi)\hat{\pi}(\psi).$$

*In particular, $\hat{\pi}$ defines naturally a bicovariant *-calculus Υ over \tilde{G} . The space $\Upsilon_{\text{inv}} = \tilde{\mathcal{A}}/\ker(\hat{\pi}) \hookrightarrow \text{im}(\hat{\pi})$ is a right $\tilde{\mathcal{A}}$ -submodule of $\tilde{\mathcal{H}}$. As a right $S(\mathbb{V})$ -module, it is generated by Γ_{inv} and $\hat{\pi}(\mathbb{V})$.*

Proof. Let us first check the consistency of (39). One method is to proceed directly, by checking the compatibility of (39) with the commutation relations (twisted commutation between \mathcal{A} and $S(\mathbb{V})$, and the kernel of the braided symmetrizer map defining $S(\mathbb{V})$) defining the algebra $\tilde{\mathcal{A}}$. Here we shall sketch a different method, by constructing first the associated bicovariant algebras.

Let $\mathcal{U} \hookrightarrow \mathcal{A} \otimes \mathcal{H}$ be a bicovariant *-algebra associated to $(\mathcal{H}, \Delta, \circ)$. The product in \mathcal{U} is given by the standard cross-product rule. The maps $\ell_{\mathcal{U}} = (\phi \otimes \text{id}): \mathcal{U} \rightarrow \mathcal{A} \otimes \mathcal{U}$ and $\varphi_{\mathcal{U}} = (\text{id} \otimes \Delta): \mathcal{U} \rightarrow \mathcal{U} \otimes \mathcal{A}$ are the corresponding left/right actions of G . Let us observe that the formula

$$da = a^{(1)} \otimes \pi(a^{(2)})$$

defines a hermitian differential $d: \mathcal{A} \rightarrow \mathcal{U}$ which is left/right-covariant, in a natural way. Let Γ be the \mathcal{A} -bimodule defined as

$$\Gamma = \left\{ \sum ad(b) \mid a, b \in \mathcal{A} \right\}.$$

By construction we see that Γ , equipped with d , is a bicovariant *-calculus over G .

In a similar way, we can construct a bicovariant *-algebra $\Pi \hookrightarrow \tilde{\mathcal{A}} \otimes \tilde{\mathcal{H}}$, equipped with the left/right actions $\ell_{\Pi}: \Pi \rightarrow \tilde{\mathcal{A}} \otimes \Pi$ and $\varphi_{\Pi}: \Pi \rightarrow \Pi \otimes \tilde{\mathcal{A}}$. The algebra \mathcal{U}

is a \tilde{G} -covariant $*$ -subalgebra of Π . Furthermore, there exists a unique differential $d: \tilde{\mathcal{A}} \rightarrow \Pi$ extending the map $d: \mathcal{A} \rightarrow \mathcal{U}$ and satisfying

$$d(\theta) = \zeta(\theta) \quad \forall \theta \in \mathbb{V}.$$

It is easy to see that the extended d is also hermitian, and left/right \tilde{G} -covariant. Let Υ be the corresponding bicovariant $*$ -calculus over \tilde{G} . We have $\Upsilon_{inv} \subseteq \tilde{\mathcal{H}}$ and $\Gamma_{inv} \subseteq \mathcal{H}$. Now we can define the extension $\hat{\pi}: \tilde{\mathcal{A}} \rightarrow \Upsilon_{inv}$ of $\pi: \mathcal{A} \rightarrow \Gamma_{inv}$ by the formula

$$\hat{\pi}(\psi) = \kappa(\psi^{(1)})d(\psi^{(2)}).$$

It is easy to see that (38) and (39) holds. Finally, properties (40) and (41) are the standard properties of the quantum germ maps. \square

Let us observe that if the group G is Γ -connected (in the sense that only the scalar elements of \mathcal{A} are annihilated by the differential d) and if there are no \varkappa -invariants in \mathbb{V} , then the restriction map $\hat{\pi}: \mathbb{V} \rightarrow \Upsilon_{inv}$ will always be injective, independently of ζ . In what follows we shall assume that the injectivity property holds. The grading on $S(\mathbb{V})$ enables us to introduce a natural filtering on Υ_{inv} , compatible with the right $S(\mathbb{V})$ -module structure. For $k \geq 0$ let us define the spaces

$$\Upsilon_{inv}^k = \hat{\pi}[S(\mathbb{V})^{k+1}] + \Gamma_{inv} \circ S(\mathbb{V})^k.$$

In general, these spaces do not form a direct sum, however they span the whole Υ_{inv} and we can introduce a filtering $\{\Upsilon_{inv}^k\}$ of Υ_{inv} by defining $\Upsilon_{inv} = \sum_{k \leq m} \Upsilon_{inv}^k$. We have $\Upsilon_{inv}^0 = \Gamma_{inv} \oplus \mathbb{V}$.

Going back to our main considerations, we see that the calculus Υ is the minimal calculus compatible with the system of maps ϱ_D and χ_E . Let us also observe that the appearance of the non-trivial filtering in Υ_{inv} is a purely quantum phenomena. By construction, the curvature/transition maps are factorizable, and can be interpreted as $\chi_E, \varrho_D: \Upsilon_{inv} \rightarrow \mathfrak{h}[\tilde{P}]$.

Lemma 3.5. *We have*

$$(42) \quad \chi_E, \varrho_D(\Upsilon_{inv}^k) \subseteq \mathfrak{hor}_P^2 \otimes S(\mathbb{V})^k + \mathfrak{hor}_P^2 \otimes S(\mathbb{V})^{k+1} \quad \forall k \in \mathbb{N} \cup \{0\}.$$

In particular, if $|k - l| \geq 2$ then $\Upsilon_{inv}^k \cup \Upsilon_{inv}^l = \{0\}$.

Proof. The above inclusions follow by applying formulae (31) and (32) together with property (35). \square

The higher-order spaces Υ_{inv}^k will be trivial only in some very special cases (including the classical case).

3.3. The Global Calculus

We can now continue with the construction of the global calculus on the quantum affine bundle \tilde{P} , following [6]. The construction builds, in an intrinsic way, a graded-differential $*$ -algebra $\Omega(\tilde{P})$, starting from $*$ -algebra $\mathfrak{h}[\tilde{P}]$, together with the system of maps $D, E: \mathfrak{h}[\tilde{P}] \rightarrow \mathfrak{h}[\tilde{P}]$ and the calculus Υ . Every covariant derivative D induces an identification

$$(43) \quad \Omega(\tilde{P}) \leftrightarrow \mathfrak{h}[\tilde{P}] \otimes \Upsilon_{inv}^\wedge = \mathfrak{vh}(\tilde{P})$$

of graded vector spaces. The graded $*$ -algebra structure of $\mathfrak{vh}(\tilde{P})$ is given by the twisted product between $\mathfrak{h}[\tilde{P}]$ and Υ_{inv}^\wedge . In accordance with [6], there exists a

natural bijection between affine covariant derivative maps $D = D_\omega$ and regular connections $\omega = \omega_D: \Upsilon_{inv} \rightarrow \Omega(\tilde{P})$ on \tilde{P} .

The elements of $\Omega(\tilde{P})$ play the role of differential forms on the quantum affine bundle \tilde{P} . There exists a natural homomorphism $\hat{H}: \Omega(\tilde{P}) \rightarrow \Omega(\tilde{P}) \hat{\otimes} \Upsilon_{inv}^\wedge$ of graded-differential $*$ -algebras, extending the map H . We have

$$\mathfrak{h}[\tilde{P}] = \hat{H}^{-1}\{\Omega(\tilde{P}) \otimes \tilde{\mathcal{A}}\}.$$

It is important to mention that our constructions of the minimal calculi Γ and Υ , as well as the global differential calculi on P and \tilde{P} , are based on the complete space of bundle derivatives. These constructions are applicable, without any change, to an arbitrary affine subspace $\mathcal{L} \subseteq \mathfrak{der}(P)$. In general, such a modified construction will give us a simpler calculus, which will be not compatible with all possible maps from $\mathfrak{der}(P)$. However, in various concrete situations even *a single derivative* $\{D\} = \mathcal{L}$ will generate the complete calculus. In classical geometry, for example, such special connections are those having *the maximal* infinitesimal holonomy (the corresponding calculus will be the classical calculus on the structure group and the bundle).

4. CONCLUDING EXAMPLES AND OBSERVATIONS

4.1. Frame Structures and Affine Bundles

In this subsection we shall briefly analyze relations between constructed affine bundles and general frame structures [5] on quantum principal bundles. Let \mathbb{V}^\wedge be the τ -exterior algebra associated to \mathbb{V} . This is a quadratic algebra given by the relations

$$\text{im}(I + \tau) \subseteq \mathbb{V} \otimes \mathbb{V},$$

and therefore it is natural to assume that $\ker(I + \tau) \neq \{0\}$. The algebra \mathbb{V}^\wedge is equipped with the induced \circ -structure.

Let us consider a quantum principal G -bundle $P = (\mathcal{B}, i, F)$. Let \mathfrak{hor}_P be a graded $*$ -algebra given by

$$\mathfrak{hor}_P \leftrightarrow \mathcal{B} \otimes \mathbb{V}^\wedge$$

at the level of graded vector spaces, with the $*$ -algebra structure specified by equalities

$$\begin{aligned} (q \otimes \vartheta)(b \otimes \eta) &= \sum_k q b_k \otimes (\vartheta \circ c_k) \eta \\ (b \otimes \vartheta)^* &= \sum_k b_k^* \otimes (\vartheta^* \circ c_k^*), \end{aligned}$$

where $\sum_k b_k \otimes c_k = F(b)$.

The elements of \mathfrak{hor}_P are interpretable as ‘frame-type’ horizontal forms on the bundle P . There exists a unique homomorphism $F^\wedge: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P \otimes \mathcal{A}$ extending both F and \varkappa . This map is coassociative and hermitian.

The constructed algebra \mathfrak{hor}_P is in the roots of the abstract definition of quantum frame structures [5]. Quite precisely, a quantum frame structure is given by a F^\wedge -covariant hermitian first-order antiderivation ∇ on \mathfrak{hor}_P such that its restriction $\overset{M}{d}: \Omega_M \rightarrow \Omega_M$ on the F^\wedge -invariants is a differential having the property that $\mathcal{V} = \Omega_M^0$ and $\overset{M}{d}(\mathcal{V})$ generate the module Ω_M^1 (and hence the whole Ω_M), and such that $\nabla(\mathbb{V}) = \{0\}$. The map ∇ plays the role of the Levi-Civita connection, and the elements of \mathbb{V}^\wedge are counterparts of the ‘coordinate forms’ on the frame bundle.

In the case of the frame structures, there exists an intrinsic choice of the translational part $\lambda \in \Lambda^1$. It is simply given by

$$(44) \quad \lambda(\theta) = \theta_{\wedge},$$

where $\theta \in \mathbb{V}$ and we have denoted by θ_{\wedge} the elements of \mathbb{V} , understood as coordinate 1-forms (the elements of \mathbb{V}^{\wedge}).

For arbitrary affine connections constructed with the help of this ‘reinterpretation’ map, Proposition 3.3 allows us to view the torsion tensor as a part of the affine curvature. Indeed, the torsion of a bundle derivative D is defined as $T(\theta) = D(\theta_{\wedge})$, as in classical geometry. Hence we have

$$(45) \quad \varrho_D(\theta) = - \sum_k \vartheta_k \varrho_D(c_k) - T(\theta),$$

which is in a complete agreement with classical geometry [11].

As a concrete illustration, let us consider the quantum Hopf fibration P . This is a quantum $U(1)$ -bundle over a quantum 2-sphere [12], defining a spin structure. The total space of the bundle P is the quantum $SU(2)$ -group [15].

The Hopf $*$ -algebra \mathcal{A} describing G is generated by a single unitary element U , corresponding to the canonical inclusion of the unit circle in the complex plane \mathbb{C} . It is equipped with the group law

$$\phi(U) = U \otimes U.$$

By definition [15], the $*$ -algebra \mathcal{B} is generated by elements α and γ together with the relations

$$\begin{aligned} \alpha\alpha^* + \mu^2\gamma\gamma^* &= 1 & \alpha^*\alpha + \gamma^*\gamma &= 1 \\ \alpha\gamma &= \mu\gamma\alpha & \alpha\gamma^* &= \mu\gamma^*\alpha & \gamma\gamma^* &= \gamma^*\gamma, \end{aligned}$$

where $\mu \in [-1, 1] \setminus \{0\}$. The fundamental representation is defined by

$$u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

and the above relations are equivalent to the unitarity of this matrix. The coproduct is given by the standard matrix rule

$$\phi(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

Let Φ be the 3-dimensional calculus over P constructed in [15]. This calculus is left-covariant and $*$ -covariant. The elements

$$\eta_3 = \pi(\alpha - \alpha^*) \quad \eta_+ = \pi(\gamma) \quad \eta_- = \pi(\gamma^*)$$

span the space Φ_{inv} . The associated right \mathcal{B} -module structure $\circ : \Phi_{inv} \otimes \mathcal{B} \rightarrow \Phi_{inv}$ is specified by

$$\begin{aligned} \mu^2\eta_3 \circ \alpha &= \eta_3 & \eta_3 \circ \alpha^* &= \mu^2\eta_3 \\ \mu\eta_{\pm} \circ \alpha &= \eta_{\pm} & \eta_{\pm} \circ \alpha^* &= \mu\eta_{\pm} \end{aligned}$$

with $\Phi_{inv} \circ \gamma = \Psi_{inv} \circ \gamma^* = \{0\}$. Here we have used the same symbols for the group entities operating on G and P .

We shall identify $G = U(1)$ with a classical subgroup of P consisting of diagonal matrices (if $\mu \neq -1, 1$ then G is precisely the classical part of P). In other words,

$$\mathcal{A} \leftrightarrow \mathcal{B}/\text{gen}(\gamma, \gamma^*).$$

More precisely $U \leftrightarrow [\alpha]$ and $U^* \leftrightarrow [\alpha^*]$. We see that the above \circ -structure on Φ_{inv} is naturally projectable down to a right \mathcal{A} -module structure on the same space (and will be denoted by the same symbol).

Let us define \mathbb{V} as the space spanned by η_{\pm} . In what follows, \mathbb{V} will be equipped with the induced \circ and $*$ -structures. Furthermore, the map $\chi: \mathbb{V} \rightarrow \mathbb{V} \otimes \mathcal{A}$ is defined as *the adjoint* action of G . Explicitly,

$$(46) \quad \chi(\eta_+) = \eta_+ \otimes U^2 \quad \chi(\eta_-) = \eta_- \otimes U^{-2}.$$

The canonical braid operator $\tau: \mathbb{V}^{\otimes 2} \rightarrow \mathbb{V}^{\otimes 2}$ has a simple form

$$(47) \quad \tau = \begin{pmatrix} 1/\mu^2 & 0 & 0 & 0 \\ 0 & 0 & 1/\mu^2 & 0 \\ 0 & \mu^2 & 0 & 0 \\ 0 & 0 & 0 & \mu^2 \end{pmatrix}$$

in the basis η_{\pm} . It follows that the τ -exterior algebra is given by the relations

$$(48) \quad \eta_{\pm}^2 = 0 \quad \eta_+ \eta_- = -\mu^2 \eta_- \eta_+.$$

These relations are just a subset of the full set of relations defining the canonical higher-order calculus over P . This calculus is given by the universal differential envelope Φ^{\wedge} of Φ ([15],[2]–Appendix B). The remaining relations are

$$(49) \quad \eta_3^2 = 0, \quad \eta_3 \eta_{\pm} = \mu^{\mp 4} \eta_{\pm} \eta_3.$$

The algebra \mathfrak{hor}_P is therefore the subalgebra of Φ^{\wedge} generated by \mathcal{B} and η_+, η_- . By construction, there exists a natural projection homomorphism $p_{hor}: \Phi^{\wedge} \rightarrow \mathfrak{hor}_P$ defined by annihilating the vertical part η_3 . Composing this projection with the differential $d: \Phi^{\wedge} \rightarrow \Phi^{\wedge}$ we obtain a first-order $*$ -antiderivation $\nabla: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$. This map gives us a canonical frame structure, and it coincides with the covariant derivative of the canonical regular connection introduced in [3]. It corresponds to the standard Levi-Civita connection on the 2-sphere (lifted to the corresponding spin bundle).

As already mentioned, in accordance with the general theory [5], the map ∇ induces a canonical differential calculus Γ on the structure group $G = \mathrm{U}(1)$. In this case the calculus is 1-dimensional, spanned by the element $\zeta = \pi(U - U^*)$. This calculus is non-classical, because the \circ -structure on Γ_{inv} is not trivial. Actually Γ is the projection of Φ on G and we have

$$\zeta \circ U = \frac{1}{\mu^2} \zeta.$$

Furthermore, the curvature map is given by

$$(50) \quad \rho_{\nabla}(U^n) = \frac{1 - \mu^{-2n}}{1 - \mu^{-2}} \rho_{\nabla}(U) \quad \forall n \in \mathbb{Z}$$

$$(51) \quad \rho_{\nabla}(U) = \mu \eta_- \eta_+ \quad \rho_{\nabla}(U^{-1}) = \mu \eta_+ \eta_-.$$

Now let us construct the affine extension \tilde{G} of G . We shall use the symbols $\xi \leftrightarrow \eta_+$ and $\xi^* \leftrightarrow \mu \eta_-$ when referring to the braided-symmetric algebra $S(\mathbb{V})$ build over \mathbb{V} . It is easy to see that the algebra $S(\mathbb{V})$ is determined by a single relation

$$(52) \quad \xi \xi^* = \mu^2 \xi^* \xi.$$

The corresponding Hopf $*$ -algebra $\tilde{\mathcal{A}}$ is determined by the unitary U generating \mathcal{A} , generators $\{\xi, \xi^*\}$ defining $S(\mathbb{V})$, and additional relations

$$(53) \quad \xi U = \frac{1}{\mu} U \xi \quad \Leftrightarrow \quad \xi^* U = \frac{1}{\mu} U \xi^*,$$

together with the coproduct specifications

$$(54) \quad \phi(U) = U \otimes U \quad \phi(\xi) = 1 \otimes \xi + \xi \otimes U^2.$$

Geometrically, the constructed quantum group is the two-fold covering of the quantum $E(2)$ group. The bundle P is interpretable as the spin 2-fold covering of the orthonormal frame bundle of the quantum sphere.

Let us naturally extend the Levi-Civita connection to the affine level, as explained at the beginning of this subsection. We are going to construct the calculus Υ , associated to the extended Levi-Civita connection ∇ . Let us assume that $\mu \in (-1, 1) \setminus \{0\}$.

Proposition 4.1. *We have*

$$(55) \quad \varrho_{\nabla}(\xi^p \xi^{*q}) = c_{pq} w \otimes \xi^p \xi^{*q} \quad \forall p, q \quad |p| + |q| > 0,$$

where

$$w = \frac{\mu^3}{1 - \mu^2} \eta_- \eta_+$$

and

$$c_{pq} = \prod_{i=1}^p \left(1 - \frac{1}{\mu^{2i+2}}\right) \prod_{j=1}^q (1 - \mu^{2j+2}).$$

In particular, it follows that the space Υ_{inv} is infinite-dimensional, with a natural basis given by the elements $\{e_{pq} \mid p, q \geq 0\}$, where

$$e_{pq} = \begin{cases} \zeta & \text{for } p = q = 0, \\ \hat{\pi}(\xi^p \xi^{*q}) & \text{otherwise.} \end{cases}$$

The vector ζ is cyclic and separating for the \circ -action of $S(\mathbb{V})$ on Υ_{inv} .

Proof. At first, let us evaluate the curvature on the generators ξ, ξ^* . In accordance with (31) and the torsionless property of ∇ we have

$$\begin{aligned} \rho_{\nabla}(\xi) &= -\xi \rho_{\nabla}(U^2) = \left(1 - \frac{1}{\mu^4}\right) w \otimes \xi \\ \rho_{\nabla}(\xi^*) &= (1 - \mu^4) w \otimes \xi^*. \end{aligned}$$

This proves the cases $(p, q) = (0, 1)/(1, 0)$. The general expression (55) follows by induction on p, q and applying the first of (36). To complete the proof, it is sufficient to observe that the image $\rho_{\nabla}(\tilde{\mathcal{A}})$ is spanned by $\rho_{\nabla}(U - U^*)$ and the elements (55). \square

If $\mu = 1$ we are in the framework of classical geometry, and all objects are classical. In particular P is the classical Hopf fibration, and Υ is the standard 3-dimensional calculus over the spin covering \tilde{G} of $E(2)$. If $\mu = -1$ the bundle P will be quantum, however the base is the classical 2-sphere. Furthermore, the calculus Υ as well as the group \tilde{G} are both non-classical. The space Υ_{inv} is 3-dimensional (spanned by ζ , $\hat{\pi}(\xi)$ and $\hat{\pi}(\xi^*)$) with $\Upsilon_{inv}^k = \{0\}$ for $k \neq 0$, and the \circ -structure is given by

$$\{\xi, \xi^*\} \circ U = -\{\xi, \xi^*\} \quad \{\xi, \xi^*\} \circ \mathbb{V} = \{0\}.$$

The braid operator τ is the standard transposition.

The presented calculations are easily incorporable in the context of arbitrary quantum frame $U(1)$ -bundles. Geometrically, such structures are interpretable as quantum Riemann surfaces. However, for general surfaces the Levi-Civita connection ∇ will generate a multi-dimensional calculus over the structure group $G = U(1)$. This phenomena will occur if and only if ρ_∇ is not an eigenvector for the right \mathcal{A} -module structure \circ on \mathfrak{zh}_P . More precisely,

Lemma 4.2. *Assume that the curvature of ∇ is non-zero. The space Γ_{inv} is naturally identifiable with $\text{im}[\rho_\nabla]$. The image of ρ_∇ is the \circ -submodule of \mathfrak{zh}_P generated by $\rho_\nabla(U)$. \square*

4.2. Affine Structures as Frame Structures

As we have seen, every bundle derivative acting on a frame bundle can be naturally extended to the affine level, so that the curvature of the extended connection appears as a combination of the original curvature (involving translational coordinates) and the torsion operator. Let us now consider the opposite problem—to construct the frame structure from a given affine connection.

Our starting point is an arbitrary abstract horizontal algebra \mathfrak{hor}_P as defined in Section 3, and we shall also assume that Ω_M is generated, as a differential algebra, by $\Omega_M^0 = \mathcal{V}$. Let us consider a map $\lambda \in \Lambda^1$. The following relations hold

$$(56) \quad -\lambda(\eta)\lambda(\theta) = \sum_k \lambda(\theta_k)\lambda(\eta \circ c_k)$$

$$(57) \quad \lambda(\theta)b = \sum_\alpha b_\alpha \lambda(\theta \circ d_\alpha),$$

as it follows from (25)–(26). Here $\varkappa(\theta) = \sum_k \theta_k \otimes c_k$ and $\sum_\alpha b_\alpha \otimes d_\alpha = F(b)$. This implies that there exists a unique homomorphism $h_\lambda: \mathcal{B} \otimes \mathbb{V}^\wedge \rightarrow \mathfrak{hor}_P$, reducing to the identity on \mathcal{B} , and extending λ . The space $\mathcal{B} \otimes \mathbb{V}^\wedge$ is equipped with the standard cross-product structure. Moreover, the map h_λ intertwines the corresponding actions of G .

Definition 3. We shall say that $\lambda \in \Lambda^1$ is *regular* iff the map h_λ is bijective.

In other words, we can identify $\mathcal{B} \otimes \mathbb{V}^\wedge$ and \mathfrak{hor}_P with the help of h_λ . If in addition there is a bundle derivative $\nabla \in \mathfrak{der}(P)$ such that $\text{im}(\lambda) \subseteq \ker(\nabla)$ then ∇ , together with the identification $\mathcal{B} \otimes \mathbb{V}^\wedge \leftrightarrow \mathfrak{hor}_P$ determines a frame structure on the bundle P . On the other hand, ∇ and λ determine an extended bundle derivative $\nabla_*: \mathfrak{h}[\tilde{P}] \rightarrow \mathfrak{h}[\tilde{P}]$. The map ∇_* contains the full information about the constructed frame structure and by construction it is the Levi-Civita connection associated to this frame structure.

In the context of the frame structures, the space $\mathfrak{h}[\tilde{P}]$ is decomposed as follows:

$$\mathfrak{h}[\tilde{P}] \leftrightarrow \mathcal{B} \otimes \underbrace{\mathbb{V}^\wedge \otimes S(\mathbb{V})}_{q\text{-plane } \Omega(\mathbb{V}, \tau)}.$$

It is interesting to observe that the marked graded $*$ -subalgebra is actually the canonical differential calculus on the quantum plane built over (\mathbb{V}, τ) . The Levi-Civita connection ∇_* reduces to the standard differential in $\Omega(\mathbb{V}, \tau)$.

4.3. Analogy With Horizontal-Vertical Decompositions

There exists an interesting formal analogy between the formalism of affine bundles \tilde{P} and differential calculus on general quantum principal bundles P . In this analogy, \mathbb{V} corresponds to the space Γ_{inv} . The analogy is more complete if we assume that the higher-order calculus on G is based on the braided exterior algebra Γ_{inv}^\vee , because $\tau \leftrightarrow -\sigma$ and $S(\mathbb{V}) \leftrightarrow \Gamma_{inv}^\vee$. Then the algebra $\mathfrak{h}[\tilde{P}]$ corresponds to the algebra $\mathfrak{vh}(P)$ of horizontally-vertically decomposed differential forms.

Let us now assume that $Q = (\mathcal{C}, i, K)$ is an arbitrary quantum principal \tilde{G} -bundle over a quantum space $M \leftrightarrow \mathcal{V}$. We can introduce a filtration on \mathcal{C} , by the formula

$$\mathcal{C}_k = K^{-1}\{\mathcal{C} \otimes \tilde{\mathcal{A}}_k\}, \quad \tilde{\mathcal{A}}_k = \left\{ \sum a\vartheta \mid a \in \mathcal{A}, \vartheta \in S(\mathbb{V})^j, j \leq k \right\}.$$

In particular, let us consider the $*$ -subalgebra $\mathcal{B} = \mathcal{C}_0$. We see that this algebra is the analog of horizontal forms in the theory of differential calculus. In other words, we have

$$K(\mathcal{B}) \subseteq \mathcal{B} \otimes \mathcal{A}.$$

Let us observe that $i(\mathcal{V}) \subseteq \mathcal{B}$ and that $P = (\mathcal{B}, i, F)$ is a quantum principal G -bundle over M . Here F is the restricted coaction map.

Definition 4. A *translaton* on \tilde{P} is every hermitian linear map $\xi: \mathbb{V} \rightarrow \mathcal{C}$ satisfying

$$(58) \quad K\xi(\theta) = \sum_k \theta_k \otimes c_k + 1 \otimes \theta,$$

where $\varkappa(\theta) = \sum_k \theta_k \otimes c_k$.

The translats are the affine-geometrical analogs of connections, as defined in the general theory. Every affine bundle \tilde{P} admits at least one translaton. The proof is similar as in the connection existence theorem [3]. It is based on the representation theory [17] of compact matrix quantum groups. The translats of \tilde{P} form a real affine space. Furthermore, it is natural to formulate

Definition 5. A translaton ξ is *regular* iff

$$(59) \quad \xi(\theta)\varphi = \sum_j \varphi_j \xi(\theta \circ d_j),$$

where $\varphi \in \mathcal{B}$, and $\sum_j \varphi_j \otimes d_j = K(\varphi)$. We shall say that ξ is *multiplicative*, iff it extends (necessarily uniquely) to a unital homomorphism $\xi: S(\mathbb{V}) \rightarrow \mathcal{C}$. Equivalently, $\xi^\otimes[\ker(Y)] = \{0\}$.

The lack of multiplicativity of ξ is measured by the values of ξ^\otimes on the generators of the ideal $\ker(Y)$. Similarly, the lack of regularity of ξ is measured by the values of the operator $\ell^\xi: \mathbb{V} \otimes \mathcal{B} \rightarrow \mathcal{C}$, defined by

$$\ell^\xi(\theta, \varphi) = \xi(\theta)\varphi - \sum_j \varphi_j \xi(\theta \circ d_j).$$

It is worth noticing that

$$(60) \quad K\ell^\xi(\theta, \varphi) = \sum_{kj} \ell^\xi(\theta_k, \varphi_j) \otimes c_k d_j$$

$$(61) \quad [\ell^\xi(\theta, \varphi)]^* = - \sum_j \xi(\theta^* \circ \kappa(d_j)^*, \varphi_j^*).$$

The operator ℓ^ξ always takes its values from the algebra \mathcal{B} . Let us fix a grade-preserving $*$ -invariant and \varkappa -covariant splitting

$$\mathbb{V}^\otimes = \ker(Y) \oplus S(\mathbb{V}).$$

This splitting induces a map $m_\xi: \mathcal{B} \otimes S(\mathbb{V}) \rightarrow \mathcal{C}$. It turns out that this map is bijective, and intertwines the corresponding actions of G . Moreover, the map m_ξ will be a $*$ -homomorphism iff ξ is regular and multiplicative.

In any case, m_ξ induces an intrinsic graded $*$ -isomorphism

$$\mathcal{B} \otimes S(\mathbb{V}) \leftrightarrow \text{gr}\{Q\},$$

which is independent of the choice of the translation ξ .

Therefore, the affine bundles introduced in a constructive way in Section 2 can be characterized as principal \tilde{G} -bundles admitting regular and multiplicative translations.

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INSTITUTO DE MATEMATICAS, UNAM, AREA DE LA INVESTIGACION CIENTIFICA, CIRCUITO EXTERIOR, CIUDAD UNIVERSITARIA, MÉXICO DF, CP 04510, MEXICO

E-mail address: micho@matem.unam.mx

<http://www.matem.unam.mx/~micho>